CNCM Online Round 4: Solutions

CNCM Administration

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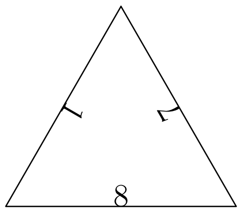
Problem 1. *A* ***triomino*** *is an equilateral triangle with side length 1 with three not necessarily distinct integers ranging from 1 to 10, inclusive, with one on each side. Triominic is laying down triominoes on a equilateral triangle shaped table with length 5, such that adjacent triominoes must have the same label on the shared sides. Let S be the number of ways Triominic can completely tile the table, given that he has a sufficient amount of each possible triomino. Find the number of positive factors of S.*



Solution. Rather than labelling the sides of each triomino, we can instead label the edges of length 1 within the larger triangle. Since there are a total of 45 edges and each edge has 10 possible labels, $S = 10^{45} = 2^{45} \cdot 5^{45}$. Thus, there are $46 \cdot 46 = \boxed{2116}$ factors.

Problem 2. *Define $\lfloor a \rfloor$ to be the largest integer less than or equal to a. It is given that*

$$\left\lfloor 100 \sum_{n=0}^{\infty} \frac{1}{2^{2^n}} \right\rfloor = 82.$$

Let S be the sum of all distinct numbers that can be expressed as $x_1 x_2$ where x_1, x_2 are distinct numbers that can be expressed as $\frac{1}{2^{2^i}}$ for nonnegative integers i. Find $\lfloor 100S \rfloor$.

Solution. Let $a_i = \frac{1}{2^{2^i}}$ and $\sum_{n=0}^{\infty} a_n = c$. We see that c^2 is $a_0(c) + a_1(c) + a_2(c) \cdots$. We can rewrite this again to

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_i a_j$$

Since we only care about the terms where $i > j$, we can remove these terms. This is done by first removing the $i = j$ terms, then dividing the whole thing by two because there is symmetry between $i > j$ and $i < j$ terms. Hence, the answer we are looking for is

$$\frac{1}{2}(c^2 - \sum_{i=0}^{\infty} a_i^2)$$

We can rewrite the inner summation.

$$\sum_{i=0}^{\infty} a_i^2 = \sum_{i=0}^{\infty} \frac{1}{2^{2^{i+1}}} = \sum_{i=1}^{\infty} \frac{1}{2^{2^i}} = \sum_{i=1}^{\infty} a_i = c^2 - a_0$$

It is easy to see that $a_0 = \frac{1}{2}$. Hence, our answer is

$$\frac{1}{2}(c^2 - c + \frac{1}{2})$$

Using our approximation of $c = 0.82$, we get

$$\frac{1}{2}(0.6724 - 0.82 + 0.5) = 0.5 \cdot 0.3524 = 0.1762$$

Hence, our answer is $\lfloor 100 \cdot 0.1762 \rfloor = \boxed{17}$

Problem 3. *$\triangle ABC$ is an equilateral triangle with side length 1. Circles ω_1, ω_2, and ω_3 have centers A, B, and C, respectively, and each of them passes through the other two vertices of the triangle.*

Construct $\triangle DEF$ such that ω_1, ω_2, and ω_3 are internally tangent to it. Let $P_1P_2P_3P_4P_5P_6$ denote the hexagon formed by the points of tangency. If the area of equilateral triangle $\triangle P_1P_3P_5$ can be expressed as $\frac{a\sqrt{b}+c}{d}$, find $1000a + 100b + 10c + d$.

Solution. Let the points be labeled in the order as below and let O be the center of ABC. Clearly $AP_3 = 1, AO = \sqrt{3}/3, \angle P_3AO = 120°$ so Law of Cosines on $\triangle AOP_3$ gives $OP_3 = \sqrt{1^2 + (\sqrt{3}/3)^2 + 2 \cdot 1 \cdot (\sqrt{3}/3) \cdot 1/2} = \sqrt{\frac{4+\sqrt{3}}{3}}$. To finish, note that $[P_1P_3P_5] = 3[P_1OP_3] = 3\left(\frac{1}{2}OP_1 \cdot OP_3 \cdot \sin\angle P_1OP_3\right) = \frac{3}{2} \cdot \sqrt{\frac{4+\sqrt{3}}{3}}^2 \cdot \frac{\sqrt{3}}{2} = \frac{4\sqrt{3}+3}{4}$. This solution was provided by **franzliszt**.

Problem 4. *Kenan plays with three towers of blocks, each with 3 stacked vertically. A "step" is when he adds one block to each tower, and then he removes all the blocks from one tower, chosen at random. He performs nine steps. The chance that at the end of each step, there is no tower that is at least 6 blocks tall can be written in the form $\frac{m}{3^n}$, where m is not divisible by three. Compute $m + n$.*

Solution.

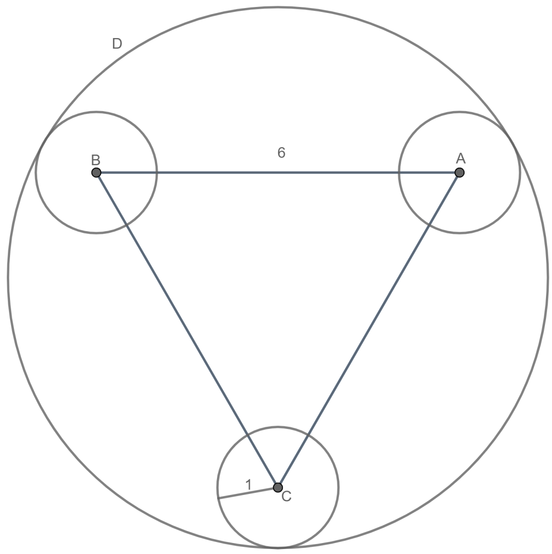
Note that the first three steps must include all three towers being removed from exactly one time. This has a $1 \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$ chance of occurring. Afterwards, there are 6 steps left and towers A, B, C have $0, 1, 2$ blocks respectively.

We proceed using complementary counting by counting how many ways there are for at least one of the towers to have 6 blocks at some point. The number of ways that tower A reaches 6 blocks is $2^6 = 64$ since all 6 steps must choose either tower B or C. The number of ways that tower B reaches 6 blocks is $2^5 \cdot 3 = 96$ since the first 5 steps must choose either tower A or C, but the final step can choose any tower. Likewise, the number of ways tower C reaches 6 blocks is $2^4 \cdot 3^2 = 144$.

Now to account for overlap, we must find the ways that exactly 2 towers reach 6 blocks. The number of ways that towers A and B reach 6 blocks is $1^5 \cdot 2 = 2$ since the first 5 steps must choose tower C and the final step can choose either tower B or C. The number of ways that towers A and C reach 6 blocks is $1^4 \cdot 2^2 = 4$ since the first 4 steps must choose tower B and the final 2 steps can choose either tower B or C. Likewise, the number of ways that towers B and C reach 6 blocks is $1^4 \cdot 2 \cdot 3 = 6$ since the first 4 steps must choose tower A, the fifth step can choose either tower A or C, and the final step can choose any tower. Finally, note that it is impossible for all 3 towers to reach 6 blocks.

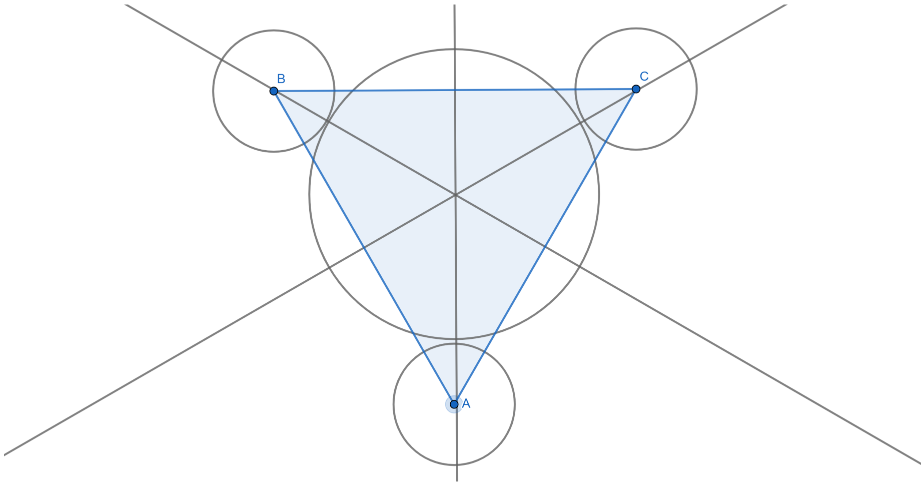
Putting this all together, we have that the number of ways for at least one of the towers to reach 6 blocks is $|A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C| = 64 + 96 + 144 - 2 - 4 - 6 + 0 = 292$ by the Principle of Inclusion Exclusion. Since there are $3^6 = 729$ ways total for the towers to be chosen, the probability that no tower reaches 6 blocks during these 6 steps is $\frac{729-292}{3^6} = \frac{437}{3^6}$. This means our final probability is $\frac{2}{9} \cdot \frac{437}{3^6} = \frac{874}{3^8}$, so our answer is $874 + 8 = \boxed{882}$.

Problem 5. *Circles A, B, C of radius 1 have centers that are pairwise 6 units apart. There is a circle D such that A, B, C are internally tangent to D. A fifth circle, E, of radius 2 is randomly drawn such that no part of E is outside of D. Let L_N be the distance from the center of circle E to the center of circle N for all $N \in \{A, B, C\}$. Let M equal max(L_A,L_B,L_C). Let P be the most likely value of M (which is not necessarily the expected value of M.) Find $\lfloor P^2 \rfloor$.*



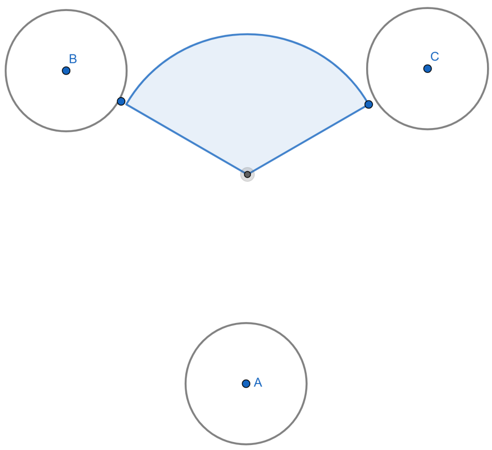
Solution. Let N' denote the center of the circle N, and define circle F to be the circle externally tangent to A, B, C - this is the circle within which E' must lie. Define $T_{X,Y}$ as the point of tangency of circles X, Y. Let O denote the center of the system. Finally, let $\triangle A'B'C'$ be oriented such that the perpendicular bisector of $\overline{BC}$ is vertical and that A is generally below B and C. (We use generally to mean that there are some components of their separation perpendicular to the downward direction.)

The next key step is to deal with the maximum function. If a point inside F is farther from A than to B or C so as to make the max function equal L_A, then it must be on the far side of the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$. Similar results hold if the maximum function is equal to L_B or L_C. In other words, if we know the region delineated by the perpendicular bisectors of $\triangle A'B'C'$ that E' is in, we know which value to take from the max.



Since E' is equally likely to be in any of the three sectors $OT_{F,A}\overset{\frown}{}T_{F,B}$, $OT_{F,A}\overset{\frown}{}T_{F,C}$, $OT_{F,B}\overset{\frown}{}T_{F,C}$, let it be WLOG in the region furthest from A which is $OT_{F,B}\overset{\frown}{}T_{F,C}$ henceforth denoted I. It is now evident that the probabilities we are dealing with are strictly infinitesimal, and furthermore that the problem could more formally be described with a probability distribution.

E' is equally likely to be in any location inside I. The probability that $\overline{A'E'}$ is of a particular length L is proportional to the length of the arc of a circle of radius L centered at A' inside I. It is clear that a particular length is immediately obvious, being the length of $\overline{A'T_{F,B}} = \overline{A'T_{F,C}}$ with associated arclength $T_{F,B}\overset{\frown}{}T_{F,C}$. Call this arc "$\lambda$". It is visually clear that lengths less than $\overline{A'T_{F,B}}$ have related arcs with length less than λ. Note that in the figure, the relevant arc is not drawn - rather, the sector shown is I.



A handwavy argument to show that arclengths of radii greater than $\overline{A'T_{F,B}}$ also have smaller parts in I is presented (and can be made rigorous.)

Consider two points H and J equidistant from A' with H closer to B and J closer to C on the boundary of I above $\overline{T_{F,B}TF,C}$. The arclength associated with the length $\overline{A'H} = \overline{A'J}$ to be considered is $\overset{\frown}{HJ}$. We seek to show that

$$\overset{\frown}{HJ} < T_{F,B}\overset{\frown}{}T_{F,C}$$

Construct two rays perpendicular to $\overline{T_{F,B}T_{F,C}}$ at $T_{F,B}, T_{F,C}$ that are directed generally away from A and call them λ_B, λ_C respectively. Let the projection of H onto λ_B be called H'' and let the projection of J onto λ_C be called J''. It is immediately clear that

$$\overset{\frown}{HJ} < \overset{\frown}{H''J''} < \overset{\frown}{T_{F,B}T_{F,C}}$$

(where all arcs are centered at A') - the latter inequality can be made rigorous by trig but is intuitively a statement that circles of larger radius have less curvature spanning the same horizontal distance and thus have less length.

Problem 6. *In the Cartesian plane, there is a hexagon with vertices at* $(-10,-10), (0,-10), (10,0), (10,10), (0,10), (-10,0)$ *in order. Four lattice points are randomly chosen such that each point is in a different quadrant, no point is outside the perimeter of the hexagon, and no point is on one of the coordinate axes. Let* A *be the expected area of the quadrilateral formed by the points in clockwise order. If* A *can be expressed as* $\frac{m}{n}$ *with* $\gcd(m,n)=1$*, compute* $m+n$*.*

Solution.

Let $(x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4)$ be the points in counter-clockwise order with (x_1,y_1) in the first quadrant, and let $(x_5,y_5)=(x_1,y_1)$. By the Shoelace Formula,

$$A = \frac{1}{2}\left|\sum_{n=1}^{4} x_n y_{n+1} + \sum_{n=1}^{4} -x_{n+1}y_n\right|$$

where A is the area of the quadrilateral. Note that x_1y_2 is positive since $x_1 > 0$ and $y_2 > 0$. Similar reasoning yields that each of the other terms is positive as well. Thus, we can rewrite the earlier formula as

$$A = \frac{1}{2}\left(\sum_{n=1}^{4} x_n y_{n+1} + \sum_{n=1}^{4} -x_{n+1}y_n\right)$$

Then by linearity of expectation,

$$\begin{aligned}
E[A] &= E\left[\frac{1}{2}\left(\sum_{n=1}^{4} x_n y_{n+1} + \sum_{n=1}^{4} -x_{n+1}y_n\right)\right] \\
&= \frac{1}{2}\left(\sum_{n=1}^{4} E[x_n y_{n+1}] + \sum_{n=1}^{4} -E[x_{n+1}y_n]\right) \\
&= \frac{1}{2}\left(\sum_{n=1}^{4} E[x_n]E[y_{n+1}] + \sum_{n=1}^{4} -E[x_{n+1}]E[y_n]\right)
\end{aligned}$$

It is easy to see that $E[x_1]=E[y_1]=-E[x_3]=-E[y_3]=\frac{11}{2}$ and we can find that $-E[x_2]=E[y_2]=E[x_4]=-E[y_4]=\frac{1}{45}\sum\limits_{n=1}^{9} n(10-n)=\frac{11}{3}$. Thus, $E[A]=\frac{1}{2}(8\cdot\frac{11}{2}\cdot\frac{11}{3})=\frac{242}{3}$ so our answer is $242+3=\boxed{245}$.

Problem 7. *Find the positive integer* a *such that* $(a+1)! \equiv a!^{13} \pmod{2a-1}$*, where* $2a-1$ *is a prime integer.*

Solution. First, we manipulate the expression:

$$a+1 \equiv a!^{12} \mod 2a-1$$

$$3a \equiv a!^{12} \mod 2a-1$$

$$3 \equiv (a-1)!^{12}a^{11} \mod 2a-1$$

We now write the expression in terms of p, a prime number, as a is not very meaningful:

$$3 \equiv \left(\frac{p-1}{2}!\right)^{12} \left(\frac{p+1}{2}\right)^{11} \mod p$$

The crux of this problem is in the usage of Wilson's Theorem, which is motivated by the existence of factorials mod a prime. Wilson's theorem says that $(p-1)! \equiv -1 \mod p$. However, we seek $\frac{p+1}{2}!$, and can get close with the realization that $(p-1)!$ can also be written as $1 \cdot (p-1) \cdot 2 \cdot (p-2) \cdots \equiv 1 \cdot -1 \cdot 2 \cdot -2 \cdots \equiv \left(\frac{p-1}{2}!\right)^2 (-1)^{\frac{p-1}{2}}$, so

$$-1 \equiv \left(\frac{p-1}{2}!\right)^2 (-1)^{\frac{p-1}{2}} \mod p$$

(In fact, this is the construction used in the proof that there exists x such that $x^2 \equiv 1 \mod p$ iff $p \equiv 1 \mod 4$.)

This trivialy reduces our expression to

$$3 \equiv \left(\frac{p+1}{2}\right)^{11} \mod p$$

yielding $\boxed{\text{p=6143 and a=3072}}$.